Stable Marriage Problem

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Synonyms: Disjoint set matching problem, Bipartite matching problem, Stable assignment problem, Deferred Acceptance Algorithm.

Keywords: Matchings, stability

1 Problem Definition

The stable marriage problem(SM)[1] is a bipartite matching problem involves a set of n men, $\{m_1, m_2, \ldots, m_n\}$ and a set of n women, $\{w_1, w_2, \ldots, w_n\}$. Each men m_i has a ordered preference list for all n women. If number of men and women are equal and each having preference list for each in opposite set this is complete preference list & matching is stable matching(SM) of size n. A matching in this content is having a set of n men-women pairs in which each man from set of men each woman from set of women appears exactly once. If a man m_i prefers woman w_k to w_j then w_k appear above w_j on man m_i's preference list and women w_i prefers man m_k to m_j then m_k appear above m_j on woman w_i's preference list.

Let denote matching by M as if a man m_i is assigned to women w_j as m_i = M(w_j) & w_j = M(m_i) as same definition hold for opposite set. For a man m_i matched partner is M(w_j). A matching is assigned such as each person(man or woman) appears exactly once in all matchings. If (m_i,w_j) $\in M$ then m_i can not have partner other than w_j & w_j can not have partner other than m_i. A pair (m_i,w_j)$\in M$ blocks a matching M, or is a blocking pair for M if the following conditions are satisfied:

1. m_i prefers w_j to M(m_i).

2. w_j prefers m_i to M(w_j).

3. Both.

A matching M is said to be stable if it admits no blocking pair. Given an instance I of SM, the problem is to find a stable matching M in I.

2 Key Results

The stable marriage problem(SM) was first defined by Gale & Shapley[1] under the name "College admission and the Stability of marriage" in American Mathematical Monthly in 1962. They showed that any instance I of SM of Size n admits at least one stable matching in polynomial time & they provided $O(n^2)$ algorithm for that. This algorithm contains a series of members of other set. Each proposal turns into either acceptance or rejection. Proposal is accepted if one prefers proposer to current partner, if proposal is accepted current partner is rejected and proposer takes place of current partner. Proposal is rejected if proposer is less preferred as compare to current partner. Algorithm can be applied in two ways,men proposing women(men oriented) or women proposing men(women oriented).

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Result: the stable matching consists of all n engagements;
set each person to be free;
while some man m is free do
    w = most preferred woman on ms list to which he has not
     yet proposed;
    if w is free then
        assign m to w;
    else
        if w prefers m to her current partner m' then
            assign m to w to be engaged and set m' to be free;
        else
            w rejects ms proposal and remains with m';
            {m remains free}
        end
    end
end
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Algorithm 1: Gale/Shapley Algorithm

Algorithm 1 shows basic man-oriented Gale/Shapley algorithm (GSA)[5]. In this algorithm men start with most preferred woman in their preference list & make proposal until they are accepted. If later they get rejected they continue proposing next women in their preference list this continues until all men and women are matched. Since total input size including preference list is $\theta(n^2)$ algorithm runs in $O(n^2)$ polynomial time.

R. W. Irving did a simple modifiction to GSA that aim to cut number of proposals to be made. Algorithm 2 shows this algorithm as Entended Gale/Shapley algorithm (EGSA)[23].

Result: the stable matching consists of all n engagements;
set each person to be free;
while *some man m is free* **do**
  w = first woman on ms list;
  if *w is engaged to* m' **then**
    set m' to be free;
  end
  assign m and w to be engaged;
  for *each successor* m'' *of m on ws list* **do**
    delete m'' from ws preference list and w from $m''s$ preference list;
  end
end

Algorithm 2: Extended Gale/Shapley Algorithm

In this algorithm(man oriented) If a woman accepts a proposal, she removes all men that are less preferred than current partner from fer preference list. With EGSA the total number of operation carried out during execution are reduced by constant time the number of pairs deleted plus number of engaged pairs. Overall worst case running time complexity remains polynomial time $O(n^2)$.

The stable matching generated by GSA is either man optimal or woman optimal depends upon proposer set, if men are proposers then it's man optimal and same for woman. A man optimal matching is matching in which man can not have a better partner in all other matchings but woman in this matching can get better partner they can not get more worse partner in any other matching so it is pessimal matching for women. When algorithm is man optimal it is woman pessimal & if we exchange roles it becomes woman optimal and man pessimal.

Hence it is natural to try to seek for a solution which is stable ans also good for both parties. There are a lot of optimization criteria for quality of stable matching that we will discuss in section Extension of SM. In applications this is inconvenient especially in a large scale matching system to include all members of other set in preference list in strict order. So we can consider two relaxations, Incomplete list and ties in their preference list. We will discuss this in section 4.

3 Applications

Algorithms for finding solutions to the stable marriage problem have applications in a variety of real-world situations, perhaps the best known of these being National Resident Matching Program(NRMP) for the assignment of graduating medical students to their first hospital appointments in US[18]. Programs like this are also exists in Japan[20], Canada[19] and Scotland[21]. Centralised matching schemes based largely on HR also occur in other practical contexts, such as school placement in New York[22]. In 2012, the Nobel Prize in Economics was awarded to Lloyd S. Shapley and Alvin E. Roth "for the theory of stable allocations and the practice of market design"[3].

An important and large-scale application of stable marriage is in assigning users to servers in a large distributed Internet service. Billions of users access web pages, videos, and other services on the Internet, requiring each user to be matched to one of (potentially) hundreds of thousands of servers around the world that offer that service. A user prefers servers that are proximal enough to provide a faster response time for the requested service, resulting in a (partial) preferential ordering of the servers for each user. Each server prefers to serve users that it can with a lower cost, resulting in a (partial) preferential ordering of users for each server. Content delivery networks that distribute much of the world's content and services solve this large and complex stable marriage problem between users and servers every tens of seconds to enable billions of users to be matched up with their respective servers that can provide the requested web pages, videos, or other services.

4 Extension of SM

4.1 SM with Incomplete preference lists

In this, preference list of each person may be Incomplete or a person may exclude some of members of preference list those(he/she) do not know or not want to be matched with. This problem is called stable marriage problem with Incomplete preference list(SMI). Here (m_i, w_j) is an acceptable pair if,m_i $'s$ preference list include w_j and w_j $'s$ preference list include m_i. In this problem number of men and women may not be same & preference list are also incomplete so it can be possible there are no perfect matching. A pair (m_i, w_j) is blocking pair for matching M if following conditions are satisfied:

1. M(m_i) $\neq$w$_j$ or M(w_j) $\neq$m$_i$ but (m_i, w_j) is an accepatable pair.
2. $w_j \succ_{m_i}$ M(m_i) or m_i is unmatched.
3. $m_i \succ_{w_j}$ M(w_j) or w_j is unmatched.

According to Gale[24] if a man/woman is paired in a matching he/she will be paired in all stable matchings or those are single will be single in all stable matchings in a given instance of SMI. This means all matchings in SMI are of same size & EGSA or GSA with slight modification can find stable matching in polynomial time[5].

4.2 SM with Ties

In this, ties are allowed in preference list. One can assign same preference to two or more person in preference list. An example of this was matching of medical students to hospital posts in UK in 2005-2006, where applicants were ranked partly based on their academic result. We denote this problem stable marriage problem with ties(SMT)[23].

In context of SMT, there are three stability notions: weak stability, strong stability and super stability[23]. They are distinguished by the definition of blocking pairs. In weak stability a blocking pair defined as (m_i, w_j) in matching M such that M(m_i) $\neq$w$_j$, $w_j \succ_{m_i}$ M(m_i), $m_i \succ_{w_j}$ M(w_j), means if both m_i & w_j prefer each other to their partner in M. In the strong stability (m_i, w_j) is blocking pair if M(m_i) $\neq$w$_j$, w_j, $w_j \succ_{m_i}$ M(m_i), $m_i \succeq_{w_j}$ M(w_j), means m_i prefers w_j to his partner in M while w_j either prefers m_i to her partner in

M. Finally in super stability (m_i, w_j) is blocking pair if M(m_i) $\neq$w$_j$, w_j, $w_j \succeq_{m_i}$ M(m_i), $m_i \succeq_{w_j}$ M(w_j), means each of m_i & w_j prefer the other to their current partner in M or indifferent between them.

Every instance of SMT admits matching in polynomial time. However SMT instance need not admit strongly stable or super stable matchings. There is a polynomial time algorithm that decides super and strongly stable matchings in $O(n^2)$ and $O(n^3)$ respectively.

4.3 SM with Ties and Incomplete preference lists

This is the generalisation of SMT and SMI that allow both Incompleteness & ties in preference list. This is called stable marriage problem with ties & incomplete preference list(SMTI). Notion of blocking pair can be defined by combining both SMT and SMI. Hence again in SMTI notion of weak, strong & super stability is defined. Let (m_i,w_j) is an blocking pair in matching M of an instance I of SMI if as follows:

- **Weak Stability:** m_i is unmatched in M or prefers w_j to his current partner in M and w_j is matched in M or prefers m_i to her current partner in M.

- **Strong Stability:** m_i is unmatched or prefers w_j to his current partner in M and w_j is unmatched in M or prefers m_i to her current partner in M or indifferent between them and vice versa by exchanging m_i and w_j.

- **Super Stability:** m_i is unmatched or prefers w_j or indifferent between them and w_j is unmatched in M or prefers m_i to her current partner in M or indifferent between them.

Like SMT, SMTI need not admit strong and super stable matchings but does admit weakly stable matching in polynomial time. These weakly stable matching can have different size as these are defined maximum & minimum weakly stable matchings for an instance of SMTI called as MAX SMTI and MIN SMTI respectively. But these problems are NP-Hard even in restrictive cases where ties appear on one set of preference lists only, the ties are at the tails of lists, there is at most one tie per list, and each tie is of length two[25]. Also , it is known that there is no polynomial time 21/19-approximation algorithm unless P=NP. (α -approximation algorithm means that it always finds a stable matching whose size is at least $1/\alpha$ fraction of the optimal size).

4.4 Gender optimal Stable Matchings

There are a lot of optimization criteria for the quality of stable matchings, but here we introduce three of them. If pair (m_i,w_j) is in a stable matching M, we define the rank of m_i in M to be the position of w_j on $m_i's$ list as $p_{m_i}(w_j)$ and the rank of w_j in M as the position of m_i on $w_j's$ list as $p_{w_j}(m_i)$.

1. **Minimum regret stable matchings:** are the stable matchings in which the rank of the worst off person is minimised. We define regret cost r(M) as
   $$\text{r(M)} = \max_(\text{m}_i, w_j) \in Mmax\{\text{p}_{m_i}(w_j),\ p_{w_j}(m_i)\} .$$
   An efficient algorithm for finding a minimum regret stable matching, given an instance of sm, is described in [13].

2. **Egalitarian stable matchings:** seek to optimise the satisfaction of both men and women simultaneously. The weight(egalitarian cost) of M is the sum of the ranks of all themen and women in M. We define egalitarian cost(e(M)) as
   $$\text{e(M)} = \sum_{(m_i,w_j)\in M} p_{m_i}(w_j) + \sum_{(m_i,w_j)\in M} p_{w_j}(m_i) .$$
   An egalitarian stable matching has minimum e(M) over all the possible stable matchings. An efficient algorithm to find an egalitarian stable matching given an sm instance, which relies heavily on the distributive lattice structure of the set of all stable matchings, is described in Irving[1987].

3. **Sex equal stable matchings:** are stable matchings in which the absolute value of the difference between the sum of the ranks of all the men and the sum of the ranks of all the women is defined this as sex equal cost(d(M)) as
   $$\text{d(M)} = \sum_{(m_i,w_j)\in M} p_{m_i}(w_j) \text{ - } \sum_{(m_i,w_j)\in M} p_{w_j}(m_i) .$$
   and this cost is minimised to find stable matchings. The problem of finding a sex-equal stable matching given an sm instance is NP-hard [14]. This was shown to be true even if the preference lists are of length at most three [16].

5 Open Problems

As noted, ties or incompleteness in preference lists may arise naturally in practical applications. In SMT & SMTI instance weak stability is the most commonly studied stability criterion, due to the guaranteed existence of such a matching. Attempting to match as many persons as possible motivates the search for large weakly stable matchings. Many approximation algorithms for finding a maximum cardinality weakly stable matching have been evolved. It remains open to find tighter upper(MAX) and lower(MIN) bounds for the approximability of this problem.

6 Future Work

As we have analysed here SM with different extensions in preference list and optimisation criteria, this motivates us for other optimization criteria as finding maximum locally stable matchings, hardness proofs and approximability results.

7 Cross Reference

- Optimal Stable Marriage
- Ranked Matching
- Stable Marriage
- Stable Marriage with Ties and Incomplete List

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